Tuesday, March 24, 2015
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## Electromagnetic Waves

Source free Maxwell $\partial_\mu F^{\mu\nu} = 0 = \partial_\mu g^{\mu\alpha} g^{\nu\lambda} (\partial_\alpha A_\lambda - \partial_\lambda A_\alpha)$

$$= \partial_\mu \partial^\mu A^\nu - \underbrace{\partial_\mu \partial^\nu A^\mu}$$

Lorentz gauge $\partial_\mu A^\mu = 0$

$$\partial_\mu \partial^\mu A^\nu = \Box A^\nu = 0$$

or $\nabla^2 A^\nu = \frac{\partial^2}{\partial t^2} A^\nu$ $\qquad \vec{\nabla} \cdot \vec{A} = \partial_\mu A^\mu + \partial_t A^0$ .

This has (among others) plane wave solutions: $A^\nu = e^{i(\vec{k}\cdot\vec{r} - \omega t)} \epsilon^\nu$

$$\nabla^2 A^\nu = -k^2 A^\nu$$
$$\frac{\partial^2}{\partial t^2} A^\nu = -\omega^2 A^\nu \quad \Rightarrow \quad k = \omega \quad \Rightarrow \quad v = 1$$

Recall: $\vec{E} = \vec{\nabla} A^0 \quad = i\vec{k} A^0$

$\vec{B} = \vec{\nabla} \times \vec{A} \quad =$

Wednesday, April 26, 2017
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# Gravitational Radiation

3 essential parts to the story:

1. Finding and solving a source free wave equation (solve Einstein's equations w/out source)
2. Detection - how do (typically weak) waves influence matter (solve geodesic equation in geometry from 1)
3. Creation - how does matter create waves (usually very involved) (solve Einstein's equations w/ source)

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Linearized GR (a setting where we have some hope of solving things)

Consider $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$ where $h_{\mu\nu}(x)$ are "small" Note: Recall "Where's Newton?"

$$g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$$

Insert, expand and keep up to linear in $h_{\mu\nu}$

Source free Einstein's equation (in trace-reverse form): $R_{\mu\nu} = 0$

First: $\delta\Gamma^\rho_{\alpha\beta} = \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\alpha h_{\beta\sigma} + \partial_\beta h_{\sigma\alpha} - \partial_\sigma h_{\alpha\beta}\right)$ since $\partial_\mu \eta_{\nu\alpha} = 0$ and would be $O(h^2)$

Recall: $R^\rho{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \underbrace{\Gamma^\rho_{\sigma\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\sigma\mu}}_{O(h^2)}$

Then: $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} = \frac{1}{2}\eta^{\rho\sigma}\left[\overset{2}{\partial_\rho \partial_\nu h_{\mu\sigma}} + \overset{3}{\partial_\rho \partial_\mu h_{\sigma\nu}} - \overset{1}{\partial_\rho \partial_\sigma h_{\nu\mu}} - \underset{4}{\partial_\nu \partial_\rho h_{\mu\sigma}} - \underset{5}{\partial_\nu \partial_\mu h_{\sigma\rho}} + \underset{6}{\partial_\nu \partial_\sigma h_{\rho\mu}}\right]$

$$= \frac{1}{2}\left[-\overset{1}{\partial_\rho \partial^\rho h_{\nu\mu}} + \overset{2}{\partial_\rho \partial_\nu h_\mu{}^\rho} + \overset{3}{\partial_\rho \partial_\mu h^\rho{}_\nu} - \underset{4}{\cancel{\partial_\nu \partial_\rho h_\mu{}^\rho}} - \underset{5}{\partial_\nu \partial_\mu h^\rho{}_\rho} + \underset{6}{\cancel{\partial_\nu \partial_\rho h^\rho{}_\mu}}\right]$$

Note:

Define: $V_\mu = \partial_\rho h^\rho{}_\mu - \frac{1}{2}\partial_\mu h^\rho{}_\rho \Rightarrow \partial_\nu V_\mu = \partial_\nu \partial_\rho h^\rho{}_\mu - \frac{1}{2}\partial_\nu \partial_\mu h^\rho{}_\rho$

Then: $R_{\mu\nu} = \frac{1}{2}\left[-\Box h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu\right] = 0$

$$\Box \equiv \partial_\rho \partial^\rho = -\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2$$

But we can do better!

Gauge freedom: We had to choose certain coordinates so that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, but we could still make coordinate changes which preserve $\eta_{\mu\nu}$, but will generally change the form of $h_{\mu\nu}$.

If we transform to $x^{\mu'} = x^\mu + \delta^\mu(x)$ w/ $\delta^\mu$ "small"
we know $g_{\mu\nu} \to g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$ which (after some work) yields:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \to g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \delta_\nu - \partial_\nu \delta_\mu \quad \text{where } \delta_\nu = \eta_{\nu\mu}\delta^\mu$$

That is we can still make gauge transformations (induced by coordinate transformations) of $h_{\mu\nu}$, i.e.

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \delta_\nu - \partial_\nu \delta_\mu \qquad \left[\text{compare to } A_\mu \to A'_\mu = A_\mu - \partial_\mu \phi\right]$$

Recall that one of the useful things about gauge freedom of potentials is that it does not change physical d.o.f. In the case at hand, the physical curvature $R^\mu{}_{\nu\lambda\rho}$ is unchanged (hence solutions to $R_{\mu\nu} = 0$ remain solutions). [Compare to $\underbrace{F_{\mu\nu}}_{\vec{E},\vec{B}} = \partial_\mu A_\nu - \partial_\nu A_\mu$]

If we choose a gauge such that $V_\mu \to V'_\mu = \partial_\gamma h^\gamma{}_\mu - \frac{1}{2}\partial_\mu h^\gamma{}_\gamma = 0$ [compare w/ $\partial_\mu A^\mu = 0$]
then we are left w/

$$R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} = 0 \quad \Rightarrow \quad \Box h_{\mu\nu} = 0 \qquad \left[\text{compare w/ } \Box A^\mu = 0\right]$$

We can immediately write down a plane wave solution: $h_{\mu\nu}(x) = a_{\mu\nu} e^{i k_\lambda x^\lambda}$

Putting this into $\Box h_{\mu\nu} = -k_\lambda k^\lambda h_{\mu\nu} = 0 \Rightarrow k_\lambda k^\lambda = 0 \Rightarrow$ null or $v = 1$ (c)

In more familiar terms: $k^\lambda = (|\vec{k}|, \vec{k})$ hence $k_\lambda k^\lambda = -|\vec{k}|^2 + \vec{k}\cdot\vec{k} = 0$

where $|\vec{k}| = \omega$, $\lambda = \frac{2\pi}{|\vec{k}|} \Rightarrow v_{wave} = \lambda \frac{\omega}{2\pi} = 1$

and wave travels in the direction of $\vec{k}$

The symmetric 4×4 matrix $a_{\mu\nu}$ describes the polarization of the wave. But even this can be cleaned up.

We still have some residual gauge freedom which changes $h_{\mu\nu}$ but does not change the Lorentz condition.

Using any $\xi_\mu$ such that $\Box \xi_\mu = 0$ w/out upsetting the story so far. But we can use these four functions to make any four components of $h_{\mu\nu}$ vanish identically.

Choosing $h_{ti} = 0$ and $\underbrace{h^\mu{}_\mu = 0}_{\text{traceless}}$ [or $a_{ti} = 0$ and $a^\mu{}_\mu = 0$] 4 terms

With this choice the Lorentz condition now becomes:

since $h_{it} = 0$

$$V_t = \partial_\rho h^\rho{}_t - \frac{1}{2}\partial_t \cancel{h^\rho{}_\rho}^{\,0} = \partial_t h^t{}_t = i\omega a_{tt} e^{ik_\lambda x^\lambda} = 0 \Rightarrow a_{tt} = 0 \quad 1 \text{ term}$$

$$V_i = \partial_\rho h^\rho{}_i - \frac{1}{2}\partial_i \cancel{h^\rho{}_\rho}_{\,0} = \partial_j h^j{}_i = i k^j a_{ji} e^{ik_\lambda x^\lambda} = 0 \Rightarrow \underbrace{k^j a_{ji} = 0}_{\text{Transverse waves!}} \quad [\text{Compare w/ } \vec{k}\cdot\vec{E} = 0]$$

If we choose $\underbrace{\vec{k} = (0,0,\omega)}_{k^\mu = (\omega,0,0,\omega)} \Rightarrow k^j a_{ji} = a_{zi} = 0$ (3 terms)

Then our 10 components of $a_{\mu\nu}$ reduce to $10 - 4 - 1 - 3 = 2$

$a_{xx} = -a_{xx}$ (traceless)

$a_{xy} = a_{yx}$ (symmetric)

This additional choice is called transverse-traceless (TT) gauge.

Finally: $$h_{\mu\nu}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik_\lambda x^\lambda}$$

Note: We can get more general solutions of the linearized theory by adding solutions of this form.
This will <u>not</u> work in full blown GR!

Thursday, April 27, 2017
7:46 AM

## Detection of gravitational waves

Now that we have a metric solution in hand we can explore how test particles respond to this (time-dependent) geometry using the geodesic equation.

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \quad \Rightarrow \quad \frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0 \quad \Rightarrow \quad \underbrace{\frac{du^\mu}{d\tau}}_{\text{"acceleration"}} = -\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta$$

for timelike and $u^\mu = \frac{dx^\mu}{d\tau}$

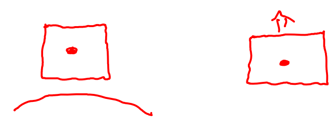
If a particle is initially at rest $u^\mu(\tau=0) = (1,0,0,0)$ then:

$$\frac{du^\mu}{d\tau}(\tau=0) = -\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\sigma}(h_{\sigma 0,0} + h_{0\sigma,0} - h_{00,\sigma}) = 0 \quad \text{since } h_{ti}=0 \;\; h_{tt}=0$$

But this means that if the particle begins at rest, then it remains at rest as the wave passes. Nothing to see here folks... move along.

But this is just a statement that the coordinate "position" of the mass is unchanged. We need something physical! How about 2 test masses and consider the invariant distance between them?

Note: We could have anticipated the need for at least 2 masses. Consider:



With one mass there is no observation to "detect" curvature.

2 does the trick!

For one mass at $x=y=z=0$ and another at $x=\varepsilon$, $y=z=0$ we find:

$$\int\sqrt{ds^2} = \int\sqrt{g_{\mu\nu}dx^\mu dx^\nu}$$
$$= \int_0^\varepsilon \sqrt{g_{xx}}\,dx$$

For $\varepsilon$ small this becomes:

$$\approx \sqrt{g_{xx}(x=0)}\;\varepsilon$$
$$= \sqrt{1+h_{xx}(x=0)}\;\varepsilon$$
$$\approx \Big[1+\frac{1}{2}\underbrace{h_{xx}(x=0)}_{a e^{ik_\lambda x^\lambda}}\Big]\varepsilon$$

so this varies w/ time!

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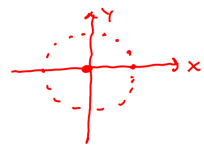
To get a clearer picture of polarization consider:

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(kz-\omega t)} \Rightarrow h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sin(kz-\omega t)$$ Travels along $z$

$a$ is small!

Then w/ $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \Rightarrow ds^2 = -dt^2 + [1+a\sin(kz-\omega t)]dx^2 + [1-a\sin(kz-\omega t)]dy^2 + dz^2$

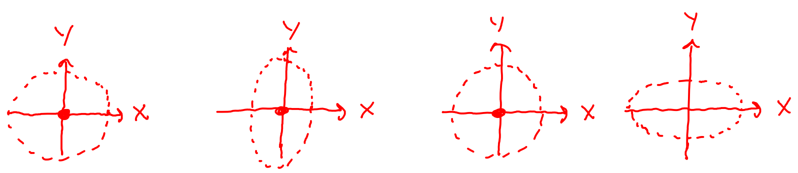
Consider a ring of masses at $z=0$ in the $x$-$y$ plane and one at the center.




The metric at $z=0$ is $ds^2\big|_{z=0} = -dt^2 + [1-a\sin\omega t]dx^2 + [1+a\sin\omega t]dy^2$

Define: $X = (1-\frac{1}{2}a\sin\omega t)x \qquad Y = (1+\frac{1}{2}a\sin\omega t)y$ Recall that $x, y$ remain constant!

Then: $ds^2\big|_{z=0} = -dt^2 + dX^2 + dY^2 + \mathcal{O}(a^2)$ We can now visualize the geometry w/ Euclidean intuition!




$\omega t = 0 \qquad \omega t = \frac{\pi}{2} \qquad \omega t = \pi \qquad \omega t = \frac{3\pi}{2}$

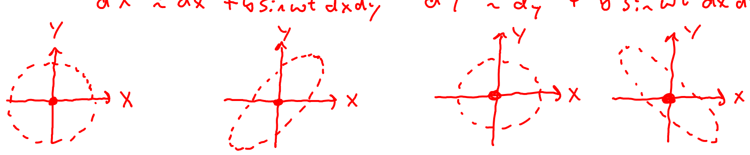
"+ polarization"

If instead we choose $h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sin(kz-\omega t) \Rightarrow ds^2\big|_{z=0} = -dt^2 + dx^2 + 2b\sin\omega t\, dx\,dy + dy^2$

Then using: $X = x + \frac{1}{2}b\sin\omega t\, y \qquad Y = y + \frac{1}{2}b\sin\omega t\, x$

$dX = dx + \frac{1}{2}b\sin\omega t\, dy \qquad dY = dy + \frac{1}{2}b\sin\omega t\, dx$

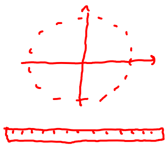
$dX^2 \approx dx^2 + b\sin\omega t\, dx\,dy \qquad dY^2 \approx dy^2 + b\sin\omega t\, dx\,dy \Rightarrow ds^2\big|_{z=0} = -dt^2 + dX^2 + dY^2$




$\omega t = 0 \qquad \omega t = \frac{\pi}{2} \qquad \omega t = \pi \qquad \omega t = \frac{3\pi}{2}$

"X polarization"

So to detect a gravitational wave you just get a ruler and measure:



But wait... wouldn't the ruler expand? Not really. Our analysis used the geodesic equation for free test particles. The bits of the ruler are not free but also experience electromagnetic binding forces which swamp the puny gravitational distortion. So yeah, a ruler will do the trick.

Well not really! You want to measure tiny displacements? You go build a big-ass interferometer.

**LIGO** (Laser Interferometer Gravitational Observatory) uses several kilometer long Michelson interferometers w/ mirrors attached to free test masses (actually hanging but free to swing). Can detect $\frac{\delta L}{L} \sim 10^{-21}$ (10 Hz – few kHz)

A 4 km arm will move $10^{-18}$ m

Oh, and they saw one!

BTW: Kip Thorne was one of the original proponents!

**LISA** (Laser Interferometer Space Antenna) will use $5 \times 10^{6}$ km long "arms" (suitable for mHz)

Pulsar Timing Arrays (Observe irregularities in what should be periodic signals from pulsars)
In this case the "arms" are parsecs long (suitable for nHz)

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## Generating gravitational waves

There are a host of issues that arise when we try to study the generation of gravitational waves. First and foremost, unlike E+M waves which we can generate uniformly by driving a system, i.e. an antenna, the mechanisms generating gravity waves of any significance are not pumped, and so will have profiles that vary in time.

Moreover, if we want to study the generation of waves, then we need to solve Einstein's equations w/ sources.

Lastly, for signals large enough to be detected (on Earth from distant sources) we need to look for large curvature effects, i.e. the linearized approach will not serve us.

All of this means that studying realistic gravity wave generation is messy. We certainly don't have time to venture into the details, but we will mention 2 interesting features.

- Just like any other form of radiation, we can take the far field limit and do a multipole expansion of the power distribution.
  For both EM and GR waves, the monopole contribution vanishes (this can be traced back to conservation of charge and mass)
  The leading term in EM is then dipole. However for GR even the dipole term vanishes (which follows from conservation of $J$ and the ability to coordinatize to zero the C.O.M. motion) so the lowest term is quadropole.

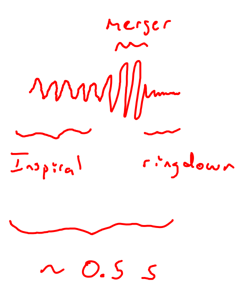
- Appreciable signals can arise from binary mergers. In particular, when massive BHs merge they can release gravitational wave energies $\sim M_\odot$. To analyze a merger, the problem is often broken up into stages. Everything could in principle be done numerically, but Einstein's equation is hard and we have to simulate over a large region to get far-field behavior.

**Inspiral**
Use post-Newtonian approximations to address 2-body problem
(use corrections to Newtonian analysis, e.g. linear approximation)

**Merger**
Numerical is about all you have.

**Ringdown** (settles to Kerr)
Use single-body BH perturbation theory.

The complete profile is often called the "chirp" characteristic of the event:




One of the fascinating things about BH mergers (compared to other merger events) is the ringdown signature. Since ringdown only happens for BHs, observation is direct observation of BH.